Studying fundamental principles in knot theory

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Introduction

In the summer of 2020, I took topology (MAT327), and for our final assignment we were asked to write an essay on a topic that matched our interests. For my topic, I studied the applications of knot theory to fluid dynamics. After completing the essay, I was left unsatisfied with my understanding of knot theory. Throughout the essay, I took many definitions for granted, without delving into the intuition behind them. My hope for this write-up is to give justification for fundamental concepts in knot theory and to deepen my knowledge of the subject.

The essay begins with a discussion of the tubular neighbourhood and the isotopy extension theorem. These tools help us decide whether we should define equivalence of knots via an isotopy or an ambient isotopy. The essay is concluded with the solution to the classical problem of showing that the trefoil knot is not the unknot. The diagrams of the trefoil and the unknot are shown in Figure 1.

Figure 1: Trefoil (on left), Unknot (on right)

Tubular Neighbourhood Theorem

Motivation for Tubular Neighbourhoods

Aside from their importance in the proof of the isotopy extension theorem, tubular neighbourhoods appear in other aspects of knot theory. An interesting example is in the application of knots to fluid dynamics, wherein flux tubes are defined by assuming the existence of a tubular neighbourhood of a knot. These flux tubes are then used to understand physical phenomena from small scale fluid vortices to the large scale behaviour of the Sun's solar flares.

A tubular neighbourhood, as the name implies, is a neighbourhood of a submanifold that looks like a tube. We restrict our focus to $\mathbb{R}^n$, as its inner product allows us to talk about vectors normal to the submanifold. However, the theorem and its proof can be extended to Riemannian manifolds, as their inner product structure resembles that of the real space.

Tubular Neighbourhood Theorem

Suppose we are given an m-dimensional submanifold M in $\mathbb{R}^n$. We define its normal space based as a point $x \in M$ as the $(n-m)$-dimensional subspace $N_xM \subseteq T_x\mathbb{R}^n$ given by

$$N_xM = \{v \in T_x\mathbb{R}^n : \langle v, w \rangle = 0, \, \forall w \in T_xM\}$$

The normal bundle $NM \subseteq T\mathbb{R}^n$ is defined as

$$NM = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M, v \in N_xM\} = \bigsqcup_{x \in M} \{x\} \times N_xM$$

We equip the normal bundle with the subspace topology, thus it inherits the hausdorff and second countable properties from its parent space $T\mathbb{R}^n$. One may also show that if M is an embedded m-dimensional submanifold in $\mathbb{R}^n$, then NM is an embedded submanifold of dimension n in $T\mathbb{R}^n$.

Note that in the definition of the normal bundle, we use the fact that $T\mathbb{R}^n$ can be identified with $\mathbb{R}^n \times \mathbb{R}^n$. Due to this identification, there is a natural way to add vectors in NM.

$$E : NM \to \mathbb{R}^n, (p, v) \mapsto p + v$$

Using these tools, we can finally define a tubular neighbourhood of a submanifold and state the tubular neighbourhood theorem.

Tubular Neighbourhood. Let M be an embedded submanifold of $\mathbb{R}^n$. A tubular neighbourhood is an open set U containing M such that U is diffeomorphic to $V \subseteq NM$ via the map E, where V is given by

$$V = \{(x, v) \in NM : |v| < \delta(x)\}$$

for $\delta : M \to \mathbb{R}$, some positive continuous function.

Tubular Neighbourhood Theorem. Let M be an embedded submanifold of $\mathbb{R}^n$. Then M has a tubular neighbourhood.

Proof. A full proof can be found in pages 139-140 of Lee's Introduction to Smooth Manifolds [1]. □

Example. To get some intuition for this theorem, consider the x-axis in $\mathbb{R}^3$, as drawn in Figure 2. Since the x-axis is a submanifold, it has a tubular neighbourhood. At point $(p,0,0)$, the normal space based at $(p,0,0)$ is the 2-dimensional plane spanned by $\{(0,1,0),(0,0,1)\}$. Taking the disjoint union of each of these planes gives the normal bundle NM. The tubular neighbourhood U is given by $U = E(V)$, where

$$V = \{(x, v) \in NM : |v| < 1\}$$

Then, $U = E(V)$ looks like a solid cylinder of radius 1 centred around the x-axis.

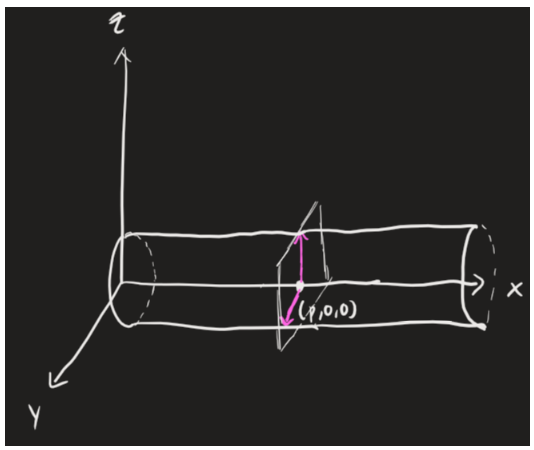

Figure 2: A Tubular Neighbourhood of the x-axis

Isotopy and Ambient Isotopy

Motivation

Let K and M be manifolds, and let $Emb(K, M)$ denote the space of embeddings from K to M. We will see that isotopy and ambient isotopy define equivalence relations on $Emb(K, M)$. Later, we shall choose between isotopy and ambient isotopy in defining a notion of knot equivalence.

Isotopy

Isotopy. An isotopy of K in M is a function $F : K \times I \to M$ such that for all $t \in I$, $F_t : K \to M$ is an embedding, where $F_t(x) = F(x, t)$.

We say that two embeddings $f : K \to M$ and $g : K \to M$ are equivalent, or isotopic, if there exists an isotopy F such that $F_0 = f$ and $F_1 = g$.

Claim. Isotopy of embeddings is an equivalence relation.

Proof. Let f, g, h be functions in $Emb(K, M)$.

An embedding $f : K \to M$ is isotopic to itself because we can just consider the isotopy $F_t = f$ for all t.

If $f \sim g$, then considering the isotopy $G_t = F_{1-t}$ shows $g \sim f$.

If $f \sim g$ and $g \sim h$, then we can find isotopies F_t and G_t such that $F_0 = f$, $F_1 = g$, $G_0 = g$, and $G_1 = h$. We piece together these isotopies as follows. Take an arbitrary real number $0 < \delta < 1$. Let ρ_0 be a bump function supported in $(-\delta, \delta)$ such that $\rho_0(0) = 1$. Similarily, let ρ_1 be a bump function supported in $(1 - \delta, 1 + \delta)$ with $\rho_1(1) = 1$. Then,

$$H_t = \begin{cases} F_{\rho_0(2t)} & 0 \le t \le \frac{1}{2} \\ G_{\rho_1(2t-1)} & \frac{1}{2} \le t \le 1 \end{cases}$$

H_t is the desired isotopy, and so isotopy of embeddings is an equivalence relation. □

Remark. *In the proof that isotopy of embeddings satisfies the transitive property, we could have chosen the isotopy H_t to be*

$$H_t = \begin{cases} F_{2t} & 0 \le t \le \frac{1}{2} \\ G_{2t-1} & \frac{1}{2} \le t \le 1 \end{cases}$$

However, as the reader might notice, this is continuous, not smooth.

Ambient Isotopy

When an isotopy $F : K \times I \to M$ satisfies the following properties, we call it an ambient isotopy.

1. $K = M$
2. $F_0 = Id_M$
3. For all $t \in I$, F_t is a diffeomorphism

An ambient isotopy can be thought of as an isotopy on the ambient space. We define an equivalence on the space $Emb(K, M)$ by saying that $f \sim g$, or f is ambient isotopic to g, if there exists an ambient isotopy F such that $F_1 \circ f = g$. Showing that this is an equivalence relation is straightforward and is left as an exercise.

Isotopy Extension Theorem

The isotopy extension theorem gives conditions in which an isotopy can be extended to the ambient space into an ambient isotopy. In the proof of the theorem, we consider the map $\hat{F} : V \times I \to M \times I$, which is given by $\hat{F}(x,t) = (F(x,t),t)$. This map is called the track of the isotopy F. By finding a vector field on the set $\hat{F}(V \times I)$, and extending it to the entire ambient space $M \times I$, the flow of the extended vector field is found to be the desired ambient isotopy.

Isotopy Extension Theorem. Let V be a compact submanifold in $M = \mathbb{R}^n$. Let $F : V \times I \to M$ be an isotopy of V. Then, F extends to an ambient isotopy with compact support.

Proof. We provide a sketch of the proof based on page 180 of Hirsch's Differential Topology [2]. Consider $\hat{F}(V \times I) \subseteq M \times I$ and construct a vector field X on this space as follows. Given a point $(x,t) \in \hat{F}(V \times I)$, we look at the arc $\hat{F}(\{x\} \times I)$ which contains this point. The arc traces the path of the point x in the submanifold V from one embedding to the next.

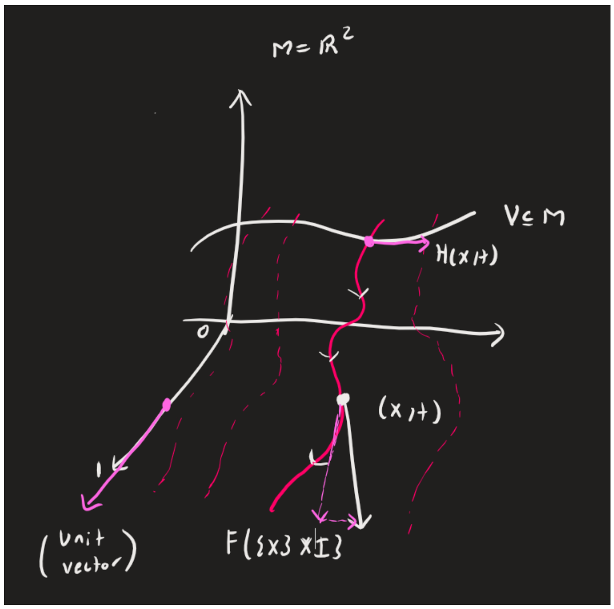

Figure 3: Constructing a Vector Field on the Track

We define the vector field X by setting $X(x,t)$ to be the tangent vector of the arc that goes through (x,t). We scale the magnitude of the tangent vector so that projecting it onto the time axis always gives a unit vector. Projecting the vector onto the manifold's axis returns what is called a time-dependent vector field. The vector is only determined by its behaviour on the manifold, so we can write X in the form $X(x,t) = (H(x,t),1)$, where $H(x,t) : \hat{F}(V \times I) \to TM$ is a time-dependent vector field. This process is visualized in Figure 3

Now, we want to extend this vector field to an open set containing $\hat{F}(V \times I)$. Hirsch is not very clear about how we actually do this, stating "[b]y means of a tubular neighbourhood ... and a partition of unity, the horizontal part of X is extended" [2]. Thus, the proof sketch from here fills in a lot of these missing details.

Fix $t = t_0$. We extend the codomain of $\hat{F}$ to $\mathbb{R}^{n+1}$. I believe the next step is to show that $\hat{F} : V \times \{t_0\} \to \mathbb{R}^{n+1}$ is an embedding, making the image an embedded submanifold. Then, we have a tubular neighbourhood U_{t_0} of $\hat{F}(V \times \{t_0\})$.

The extension of the vector field X onto this tubular neighbourhood is obtained by a "copy-paste" process, as shown in Figure 4. The first component of X, being $H(x,t)$ is extended to the tubular neighbourhood, and this extension is called $\tilde{H}(x,t)$.

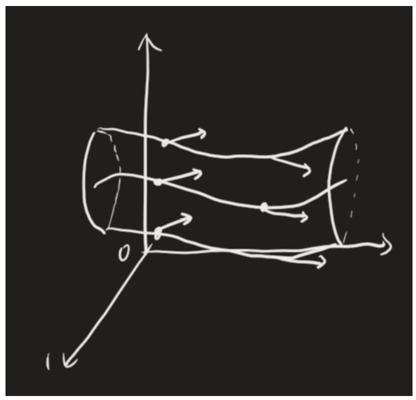

Figure 4: Copy-Paste of $H(x,t)$

Repeating this process for all $t_0 \in I$ gives a collection of tubular neighbourhoods U_t and extended vector fields. The $\{U_t \cap I\}$ form an open cover of $\hat{F}(V \times I)$ in $M \times I$. Let $\{\rho_t\}$ be a partition of unity subordinate to this cover. Define

$$\tilde{\tilde{H}}(x,t) = \sum \rho_t \tilde{H}(x,t)$$

$\tilde{\tilde{H}}(x,t)$ is a time-dependent vector field from $U = \bigcup U_t$ to TM. We extend the domain of this time-dependent vector field to $M \times I$ using a bump function, and call this extension G. Such a $G : M \times I \to TM$ is a compactly supported time-dependent vector field, and its flow generates the desired ambient isotopy. $\square$

Knots

We now propose a definition of a knot. An attempt at a definition would be to say that a knot is a topological embedding of S^1 into $\mathbb{R}^3$. This makes intuitive sense, but such a definition allows for the existence of knots that behave in wild ways. Figure 5 displays such a knot, wherein the knotted part occurs infinite times and becomes infinitely small.

Figure 5: A Wild Knot

To avoid this messy behaviour, we say that a knot is tame if whenever we draw a ball around a piece of it, the knot appears as a straight line going through the ball, as shown in Figure 6. More formally, a knot K is tame if $\forall p \in K$ there exists a neighbourhood U of p such that U is homeomorphic to a ball of radius 1 and $U \cap K$ is homeomorphic to the a straight line.

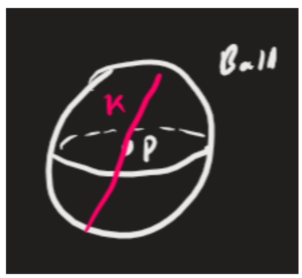

Figure 6: Tame Knots

Remark. *Unlike most subjects in mathematics, where further generalzation and abstraction is preferred, knot theory is a field that remains within the world of $\mathbb{R}^3$. As well, we only care about tame knots, and ignore wild ones such as those in Figure 5. I recently heard a talk by Dror Bar-Natan about extending the definitions of knots to $\mathbb{R}^4$, and visualizing their 3-dimensional cross-sections. At this talk, I asked him if there are any definitions of n-knots, and he responded that there are none that he knows of.*

We define a smooth, or C^∞ knot, as follows.

Smooth Knots. A smooth knot is the image of an embedding of S^1 into $\mathbb{R}^3$.

Claim. A smooth knot is tame.

Proof. Let K be a smooth knot given by $f : S^1 \to \mathbb{R}^3$ an embedding. Let $p \in K$ be an arbitrary point. Since f is an immersion, we can apply the immersion theorem near $f^{-1}(p)$, the point that gets mapped to p. There exists a chart (U, Φ) near $f^{-1}(p)$ in S^1 and a chart (V, Ψ) near p in $\mathbb{R}^3$ such that the transition map looks like

$$\Phi \circ f \circ \Psi^{-1} : x \mapsto (x, 0, 0)$$

Choose $\epsilon > 0$ so that $B_\epsilon(p)$ is an epsilon-ball centred at p and is contained in V. The image of $B_\epsilon(p) \cap K$ is a line, giving a homeomorphism between $B_\epsilon(p) \cap K$ and a line. As well, $B_\epsilon(p)$ is homeomorphic to a ball of radius 1. Thus, smooth knots are tame. $\square$

Hereinforth, we refer to smooth knots as knots, and use them as our definition of a knot. There are other definitions that one may find in standard texts. We could have called a knot a smooth, injective, piecewise-linear (PL) map from S^1 to $\mathbb{R}^3$. We may have also defined it by saying a knot is simply its knot diagram. The former definition is known as the PL-category of knots, and the latter definition makes the combinatorial category of knots. Showing that the smooth, PL and combinatorial definitions are equivalent is a tedious exercise involving the tubular neighbourhood theorem. I was initially planning on doing this exercise as my essay topic, but decided against that after recognizing how long it would take.

Knot Equivalence

We need a notion of equivalence between two knots. Let K_1 and K_2 be two knots, where K_1 is the image under the embedding of $f : S^1 \to \mathbb{R}^3$, and K_2 is the image under the embedding $g : S^1 \to \mathbb{R}^3$.

One possible equivalence is to say $K_1 \sim K_2$ if f is homotopic to g. That is, there exists a continuous map $H : S^1 \times I \to \mathbb{R}^3$ such that $H_0 = f$ and $H_1 = g$. The issue with this method of equivalence is that we can just stretch knots until they vanish, making all knots the unknot, as shown in Figure 7.

In our second attempt at a definition, since knots are defined using functions in $Emb(S^1, \mathbb{R}^3)$, we say that two knots are equivalent if their corresponding embeddings are isotopic. However, even isotopy is too weak, as it gives us the power to throw the knotted components of a knot away to infinity. The following example makes this clear.

Example. Consider the knot in Figure 8, and call it L. In this example, we abuse our definition of a knot and assume that L is given by an embedding of some function from $\mathbb{R}$ to $\mathbb{R}^3$. Suppose, as a set, L is the x-axis everywhere except for $x \in (-\epsilon, \epsilon)$, where it is given by some set $K \subseteq \mathbb{R}^3$.

The idea in solving this problem is to define an isotopy $F_t : \mathbb{R} \to \mathbb{R}^3$ which is the x-axis everywhere except on $(-\epsilon + \frac{1}{1-t}, \epsilon + \frac{1}{1-t})$, where it is K. The image of F_1 would then be the x-axis, because the knotted component K gets pushed off to infinity.

As the above example shows, isotopy is too weak to use as an equivalence between knots. We strengthen our definition as follows

Equivalence of Knots. Suppose two knots K and $\tilde{K}$ are given by embeddings f and g respectively. We say they are equivalent if f is ambient isotopic to g.

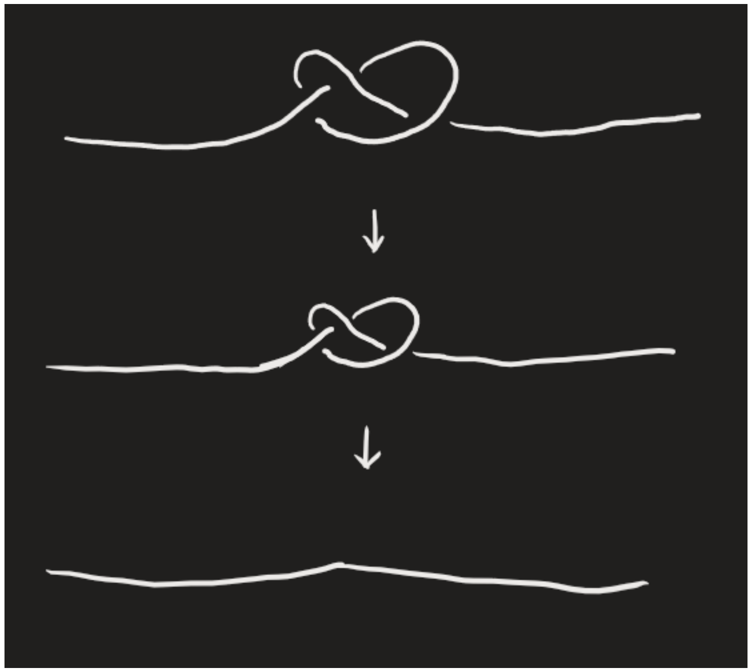

Figure 7: Stretching a Knot via Homotopy

Figure 8: Rolling a Knot to Infinity

Remark. *It remains to be shown that the definition of equivalence of knots is well defined. That is, given two embeddings f and g that parametrize the same knot K, f and g are ambient isotopic. I was unable to figure out a proof of this claim.*

Knot Groups

In this section, we take a slight detour and talk about a concept in topology known as the fundamental group. As well, we define the knot group, which is a property of knots that is invariant under ambient isotopy. Given a topological space X, its fundamental group based at a point $p \in X$, and denoted $\Pi_1(X, p)$, is defined by

Fundamental Group of X based at p. The equivalence classes of path-homotopy on the loops based at p.

If we add the assumption that X is path connected, then the fundamental group does not depend on the choice of starting point, and is denoted $\Pi_1(X)$. We also define the knot group as follows.

The Knot Group. Given a knot K, its knot group is the fundamental group of $\mathbb{R}^3 - K$.

Example. The knot group of the unknot U is computed in Figure 9, and is determined to be isomorophic to $\mathbb{Z}$. Since $\mathbb{R}^3 - U$ is a path-connected space, it suffices to consider loops based at the origin. Null-homotopic loops are those that do not cross through the ring made by the unknot U. The remaining equivalence classes of loops are made by going through the ring an integer number of times, where negative integers correspond to passing through the ring in the opposite direction.

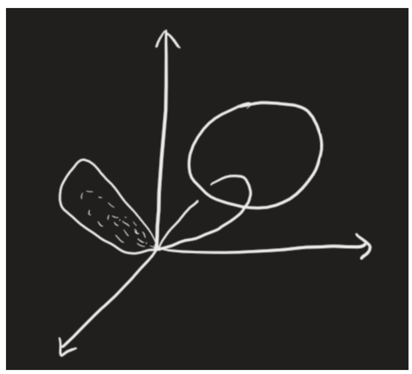

Figure 9: Loops in $\mathbb{R}^3 - U$

There are algorithmic ways to compute the knot group, such as via its Wirtinger presentation. Rather than delve into the theory of these computations, we take it as a given that the knot group of the trefoil knot is non-abelian. There is a visual method I came across which computes the trefoil knot's group by considering its projection onto the torus, as seen in [7], but I do not understand how Van Kampen's theorem is used in the calculation.

The Knot Problem

The objective of knot theory is to determine whether a knot is equivalent to the unknot via a sequence of deformations, and without performing any cutting or gluing. In mathematical language, deforming a knot without cutting or gluing it is done via an ambient isotopy. With definitions out of the way, we now solve the classical problem of whether the trefoil knot is equivalent to the unknot.

From physical intuition, we already know the answer; they are not equivalent. The reader can build a trefoil knot and spend days trying to turn it into an unknot without success. However, we need to come up with a proof of this, as there might exist a sequence of deformations that we have yet to try. The proof technique uses a principle known as knot invariants.

A knot invariant is a property of a knot that remains unchanged under ambient isotopy. We will now verify that the knot group is an invariant of the knot.

Claim. The knot group is a knot invariant.

Proof. Let K and $\tilde{K}$ be knots given by embeddings f and g respectively. Suppose f and g are ambient isotopic via the ambient isotopy $F : \mathbb{R}^3 \times I \to \mathbb{R}^3$. The map F_1 is a diffeomorphism with the property that $F_1(K) = \tilde{K}$. Thus, we can restrict F_1 to a diffeomorphism between $\mathbb{R}^3 - K$ and $\mathbb{R}^3 - \tilde{K}$. Since these spaces are diffeomorphhic, they must have the same fundamental group, which is as desired. □

We now conclude the essay with the proof that the Trefoil knot is different from the Unknot.

Claim. There does not exist an ambient isotopy between the trefoil knot and the unknot, meaning they are not equivalent.

Proof. Suppose there did exist an ambient isotopy between the trefoil knot and the unknot. Then, they would need to have isomorphic knot groups. However, we know that this is not true, as one group is abelian while the other is non-abelian, giving a contradiction. □

We can actually prove something even stronger using the isotopy extension theorem.

Claim. There does knot exist an isotopy between the trefoil knot and the unknot.

Proof. If we could find an isotopy between these two knots, since S^1 is compact, this isotopy would extend to an ambient isotopy. Applying the previous theorem would give the same contradiction. □

Remark. *The reader might recall the example where we used an isotopy to roll a knot to infinity, making it equivalent to the x-axis, as shown in Figure 8. The reason we cannot extend this isotopy to an ambient isotopy is because we assumed that the domain space of the embedding is $\mathbb{R}$, a non-compact space.*

We have finally shown that the unknot cannot be equivalent to the trefoil via both an isotopy and ambient isotopy. If the reader is interested, a connection can be made between isotopy/ambient isotopy and something known as Reidemeister moves. In summary, Reidemeister moves are the ways we can deform a knot diagram, and it can be shown that deforming a knot diagram is no different than performing an isotopy/ambient isotopy (depending on which moves you permit).

References

[1] John Lee. *Introduction to Smooth Manifolds.* Springer, New York, New York, 2013.

[2] Morris W Hirsch. *Differential Topology.* Springer, New York, New York, 1976.

[3] Raymond WB Lickorish. *An Introduction to Knot Theory.* Springer, New York, New York, 1997.

[4] ND Gilbert and T Porter. *Knots and Surfaces.* Oxford University Press, New York, New York, 1994.

[5] Dror Bar-Natan's Knot Theory Lectures (can be found on `http://www.math.toronto.edu/~drorbn/classes/20-1350-KnotTheory/`

[6] `https://math.ucr.edu/~res/math260s10/isotopyextension.pdf` (Course notes I found online)

[7] `https://math.stackexchange.com/questions/1774198/fundamental-group-of-mathbbr3-minus-trefoil-knot?noredirect=1&lq=1/`

Closing Remarks

Thank you for the great summer semester! If you are taking a year off from school, I hope you have a restful break.

Figure 10: Me because this class is ending